Isospin Violation
and the Magnetic Moment of the Muon[*]

V. Cirigliano, G. Ecker and H. Neufeld

Institut für Theoretische Physik, Universität Wien
Boltzmanngasse 5, A-1090 Vienna, Austria

Abstract

We calculate the leading isospin-violating and electromagnetic corrections for the decay $\tau^- \to \pi^0 \pi^- \nu_\tau$ at low energies. The corrections are small but relevant for the inclusion of τ decay data in the determination of hadronic vacuum polarization especially for the anomalous magnetic moment of the muon. We show that part of the systematic differences between the measured form factors in $\tau^- \to \pi^0 \pi^- \nu_\tau$ and $e^+ e^- \to \pi^+ \pi^-$ is due to isospin violation.

1. Precise knowledge of hadronic vacuum polarization is essential for a reliable determination of both the running of the QED fine structure constant and of the anomalous magnetic moment of the muon a_μ. For the latter, the low-energy structure of hadronic vacuum polarization is especially important. In fact, about 70 % of $a_\mu^{\rm vacpol}$ is due to the two-pion intermediate state for $4M_\pi^2 \leq t \leq 0.8~{\rm GeV}^2$ (see, e.g., Ref. [1]).

A precision of 1 % has been achieved in the calculation of $a_\mu^{\rm vacpol}$ by including [2] the more accurate τ decay data [3] in addition to experimental results for $\sigma(e^+ e^- \to {\rm hadrons})$. This is possible because of a CVC relation between electromagnetic and weak form factors in the isospin limit. However, both the aforementioned theoretical accuracy and the new high-precision experiment at Brookhaven [4] warrant a closer investigation of isospin violation. A crucial quantity in this connection is the pion mass difference $M_{\pi^+}^2 - M_{\pi^0}^2 = 0.067~\overline{M_\pi^2}$ that is almost exclusively due to electromagnetic effects. Therefore, both the light quark mass difference and electromagnetism have to be taken into account in a consistent treatment of isospin violation.

We concentrate in this letter on isospin violation in the reactions $\tau^- \to \pi^0 \pi^- \nu_\tau$ and $e^+ e^- \to \pi^+ \pi^-$ at low energies. Chiral perturbation theory (CHPT) [5] is the only framework where such corrections can be reliably calculated for the standard model in a systematic low-energy expansion. More specifically, we are going to calculate the leading corrections of both $O[(m_u - m_d)p^2]$ and $O(e^2 p^2)$ for the CVC relation between the two-pion (vector) form factors in the two processes. A systematic chiral counting will be essential to extract the leading effects at low energies.

2. The contribution of hadronic vacuum polarization at $O(\alpha^2)$ to the anomalous magnetic moment of the muon $a_\mu = (g_\mu - 2)/2$ is given by [6]

$$a_\mu^{\rm vacpol} = \frac{1}{4\pi^3} \int_{4M_\pi^2}^\infty dt K(t) \sigma_{e^+ e^- \to {\rm hadrons}}^0 (t) \tag{1}$$

where $K(t)$ is a smooth kernel concentrated at low energies. The superscript in $\sigma_{e^+ e^- \to {\rm hadrons}}^0$ denotes the "pure" hadronic cross section with QED corrections removed [7]. For the two-pion final state under discussion this means that $F_V(t)$ in

$$\sigma_{e^+ e^- \to \pi^+ \pi^-}^0 (t) = \frac{\pi \alpha^2 \beta_{\pi^+ \pi^-}^3 (t)}{3t} |F_V(t)|^2 \tag{2}$$

$$\beta_{\pi^+ \pi^-}(t) = \lambda^{1/2}(1, M_{\pi^+}^2/t, M_{\pi^+}^2/t) = (1 - 4M_{\pi^+}^2/t)^{1/2}$$

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx) \tag{3}$$

is the vector form factor of the pion with QED turned off (except for electromagnetic contributions to the charged meson masses).

The decay $\tau^- \to \pi^0 \pi^- \nu_\tau$ is in general governed by two form factors f_+, f_-. In the absence of electromagnetic corrections, these form factors are functions of the single variable t that is again the invariant mass squared of the two pions in the final state. The inclusion of electromagnetic effects generates shifts to the form factors which depend on a second Dalitz variable $u = (P_\tau - p_{\pi^-})^2$. Denoting by $f_+(t), f_-(t)$ the u-independent

components of the form factors (to be defined precisely below), the decay distribution with respect to t takes the general form

$$\frac{d\Gamma(\tau^- \to \pi^0\pi^-\nu_\tau)}{dt} = \frac{\Gamma_e^{(0)} S_{\text{EW}} |V_{ud}|^2}{2m_\tau^2} \beta_{\pi^0\pi^-}(t) \, (1 - \frac{t}{m_\tau^2})^2 \left\{ |f_+(t)|^2 \left[(1 + \frac{2t}{m_\tau^2})\beta_{\pi^0\pi^-}^2(t) \right. \right.$$
$$\left. \left. + \frac{3\Delta_\pi^2}{t^2} \right] + 3|f_-(t)|^2 - 6\text{Re}\,[f_+^*(t)f_-(t)]\frac{\Delta_\pi}{t} \right\} G_{\text{EM}}(t) \qquad (4)$$

with

$$\Gamma_e^{(0)} = \frac{G_F^2 m_\tau^5}{192\pi^3} \;,\; \Delta_\pi = M_{\pi^+}^2 - M_{\pi^0}^2 \;,\; \beta_{\pi^0\pi^-}(t) = \lambda^{1/2}(1, M_{\pi^0}^2/t, M_{\pi^+}^2/t) \;.$$

The factor S_{EW} takes into account the dominant short-distance electroweak corrections [8]. In the discussion of semi-leptonic τ decays, the QED scale of S_{EW} is usually chosen at the τ mass. Thus, to lowest order in α, the short-distance enhancement factor is given by $S_{\text{EW}} = 1 + (\alpha/\pi)\log(M_Z^2/m_\tau^2)$. Including the dominant electromagnetic higher-order effects, one finds the commonly used value [2] $S_{\text{EW}} = 1.0194$. The factor $G_{\text{EM}}(t)$ arises from the integration of the u-dependent electromagnetic correction over the Dalitz variable u. In principle, the spectrum distortion $G_{\text{EM}}(t)$ receives both virtual and real photon contributions. The measured electronic decay rate of the τ lepton is related to $\Gamma_e^{(0)}$ by [9]

$$\Gamma(\tau^- \to e^-\bar{\nu}_e\nu_\tau(\gamma)) = \Gamma_e^{(0)} \left[1 + O(\frac{m_e^2}{m_\tau^2}) \right] \left[1 + \frac{\alpha}{2\pi}(\frac{25}{4} - \pi^2) + O(\alpha^2) \right] \;. \qquad (5)$$

In the isospin limit[1] we have $M_{\pi^+} = M_{\pi^0}$, $S_{\text{EW}} = G_{\text{EM}}(t) = 1$ and

$$f_+(t) = F_V(t)$$
$$f_-(t) = 0 \;, \qquad (6)$$

implying the CVC relation

$$\sigma^0_{e^+e^- \to \pi^+\pi^-}(t) = \frac{1}{\mathcal{N}(t)\,\Gamma_e^{(0)}} \frac{d\Gamma(\tau^- \to \pi^0\pi^-\nu_\tau)}{dt}$$
$$\mathcal{N}(t) = \frac{3|V_{ud}|^2}{2\pi\alpha^2 m_\tau^2} t(1 - \frac{t}{m_\tau^2})^2 \, (1 + \frac{2t}{m_\tau^2}) \;. \qquad (7)$$

Including isospin violation to leading order, $O[(m_u - m_d)p^2]$ and $O(e^2p^2)$, we find from Eq. (4) that still only the form factor $f_+(t)$ survives to this order. The modified CVC relation takes the form

$$\sigma^0_{e^+e^- \to \pi^+\pi^-}(t) = \frac{1}{\mathcal{N}(t)\,\Gamma_e^{(0)}} \frac{d\Gamma(\tau^- \to \pi^0\pi^-\nu_\tau)}{dt} \frac{R_{\text{IB}}(t)}{S_{\text{EW}}} \qquad (8)$$

$$R_{\text{IB}}(t) = \frac{1}{G_{\text{EM}}(t)} \frac{\beta_{\pi^+\pi^-}^3(t)}{\beta_{\pi^0\pi^-}^3(t)} \left| \frac{F_V(t)}{f_+(t)} \right|^2 \;. \qquad (9)$$

---

[1]From now on, the term isospin limit stands for both $m_u = m_d$ and $e = 0$.

Bremsstrahlung of soft photons (in principle contained in the function $G_{\rm EM}(t)$) is subtracted (at least in some approximation to be discussed below) directly from the raw data [3]. In the analysis of Ref. [2], some additional isospin-violating corrections such as the width difference $\Gamma_{\rho^+} - \Gamma_{\rho^0}$ were applied. The importance of the phase space correction factor $\beta^3_{\pi^+\pi^-}(t)/\beta^3_{\pi^0\pi^-}(t)$ has very recently been emphasized in Ref. [10].

It is the purpose of this work to estimate the remaining contributions to $R_{\rm IB}(t)$. Working at leading order, the form factor $F_V(t)$ needs to be calculated to $O[(m_u - m_d)p^2]$ with physical meson masses (but without explicit photonic corrections) whereas $f_+(t)$ must be calculated to both $O[(m_u - m_d)p^2]$ and $O(e^2 p^2)$ if $d\Gamma/dt$ is to be extracted from actual τ decay data.

3. To first order in isospin breaking and to first non-trivial order in the low-energy expansion, isospin violation manifests itself in the pion vector form factor $F_V(t)$ only in the masses of the particles contained in the loop amplitude:

$$F_V(t) = 1 + 2H_{\pi^+\pi^-}(t) + H_{K^+K^-}(t) \tag{10}$$

with [11]

$$H_{PQ}(t) = \frac{1}{F^2}\left[h_{PQ}(t,\mu) + \frac{2}{3}t L_9^r(\mu)\right] , \tag{11}$$

where F denotes the pion decay constant in the chiral limit. The expression for the loop function $h_{PQ}(t,\mu)$ is reported in the Appendix. The low-energy constant $L_9^r(\mu)$ governs the charge radius of the pion which is in turn completely dominated by the ρ resonance. We will use the prescription of Ref. [12] where the CHPT form factor (10) of $O(p^4)$ was matched to the resonance region:

$$F_V(t) \;\; = \;\; \frac{M_\rho^2}{M_\rho^2 - t - iM_\rho\Gamma_\rho(t)} \exp\left[2\tilde{H}_{\pi^+\pi^-}(t) + \tilde{H}_{K^+K^-}(t)\right] , \tag{12}$$

with the hadronic off-shell width (for the present case of the ρ^0, the charged pion and kaon masses must be inserted)

$$\Gamma_\rho(t) = \frac{M_\rho t}{96\pi F_\pi^2}\left[\beta^3_{\pi\pi}(t)\theta(t - 4M_\pi^2) + \frac{1}{2}\beta^3_{KK}(t)\theta(t - 4M_K^2)\right] \tag{13}$$

and with a subtracted loop function (setting $\mu = M_\rho$)

$$\tilde{H}_{PQ}(t) = \frac{{\rm Re}\, h_{PQ}(t, M_\rho)}{F^2} . \tag{14}$$

The representation (12) has the following attractive features [12]:

- By construction, it has the correct low-energy behaviour to $O(p^4)$ and its asymptotic behaviour is in accordance with QCD;

- It gives an excellent description of $e^+e^- \to \pi^+\pi^-$ data up to $t \sim 1$ GeV^2 with the single parameter $M_\rho \simeq 775$ MeV.
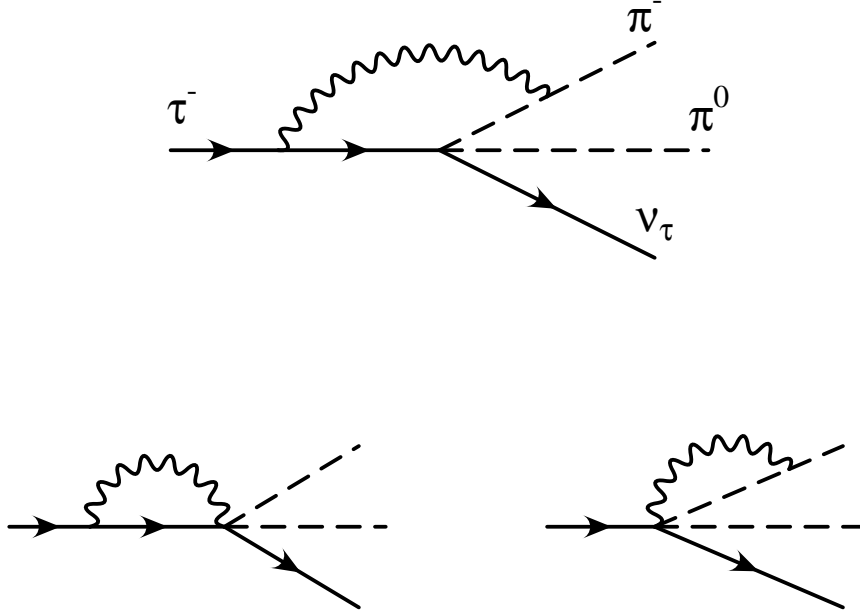
Figure 1: Photon loop diagrams (without wave function renormalization).

For our present purposes, the representation (12) exhibits in addition the correct behaviour to first order in isospin violation. To the order we are working, neither the $\rho^+ - \rho^0$ mass difference nor isospin-violating corrections to F_π (we use $F = F_\pi = 92.4$ MeV) enter. All the isospin violation to this order is contained in the physical meson masses in $\Gamma_\rho(t)$ and $\tilde{H}_{PQ}(t)$ and this feature will carry over to the form factor $f_+(t)$, except for additional purely electromagnetic corrections.

At our level of precision, $\rho - \omega$ mixing does not appear either. Such higher-order effects are not necessarily negligible numerically (see, e.g., Ref. [13]). They can be and partly are taken into account in the actual analysis of the data (see, e.g., Ref. [2]) or can be included in the theoretical error of a_μ^{vacpol}.

4. To first order in isospin violation, this time including explicit photonic corrections, the form factor $f_+(t, u)$ is given by

$$f_+(t, u) = 1 + 2H_{\pi^0\pi^-}(t) + H_{K^0K^-}(t) + f_{\text{loop}}^{\text{elm}}(u, M_\gamma) + f_{\text{local}}^{\text{elm}} . \tag{15}$$

Compared to the form factor $F_V(t)$ in (10), the appropriate meson masses appear in the loop amplitude and there is an additional electromagnetic amplitude containing both the photon loop diagrams shown in Fig. 1 and an associated local part. The electromagnetic amplitude depends on the second Dalitz variable $u = (P_\tau - p_{\pi^-})^2$ but not on t. Using a small photon mass M_γ as infrared regulator, the electromagnetic amplitudes are given by

(the corresponding calculation for K_{l3} will be presented in Ref. [14] with more details)

$$f_{\text{loop}}^{\text{elm}}(u, M_\gamma) = \frac{\alpha}{4\pi}\Big[(u - M_\pi^2)\,\mathcal{A}(u) + (u - M_\pi^2 - m_\tau^2)\,\mathcal{B}(u)$$

$$+2\,(M_\pi^2 + m_\tau^2 - u)\,\mathcal{C}(u, M_\gamma)\Big] \tag{16}$$

$$f_{\text{local}}^{\text{elm}} = \frac{\alpha}{4\pi}\left[-\frac{3}{2} - \frac{1}{2}\log\frac{m_\tau^2}{\mu^2} - \log\frac{M_\pi^2}{\mu^2} + 2\log\frac{m_\tau^2}{M_\rho^2} - (4\pi)^2\left(-2K_{12}^r(\mu) + \frac{2}{3}X_1 + \frac{1}{2}\tilde{X}_6^r(\mu)\right)\right]. \tag{17}$$

Expressions for the functions $\mathcal{A}(u)$, $\mathcal{B}(u)$, and $\mathcal{C}(u, M_\gamma)$ are given in the Appendix. We have included logarithmic terms in $f_{\text{local}}^{\text{elm}}$ to make it scale independent. The coupling constants K_{12}, X_1, and X_6 appear in the low-energy expansion of the standard model at order e^2p^2 with inclusion of virtual photons [15] and of both virtual photons and leptons [16]. Here we have pulled out the short-distance part X_6^{SD} of X_6 by the decomposition [14]

$$X_6^r(\mu) = X_6^{\text{SD}} + \tilde{X}_6^r(\mu) \tag{18}$$

where

$$e^2 X_6^{\text{SD}} = -\frac{e^2}{4\pi^2}\log\frac{M_Z^2}{M_\rho^2} = 1 - S_{\text{EW}} - \frac{e^2}{4\pi^2}\log\frac{m_\tau^2}{M_\rho^2} \; . \tag{19}$$

The size of these contributions is discussed in the next section.

The loop function $f_{\text{loop}}^{\text{elm}}(u, M_\gamma)$ encodes universal physics related to the Coulomb interaction between the τ and the charged pion. In other words, the u-dependence of the loop amplitude has little to do with low-energy QCD and thus with the chiral expansion. Rather, it represents the contribution to the loop integral given by low-energy virtual photons (the ultraviolet part being absorbed in the definition of the local amplitude). It is therefore natural to factorize these universal effects in an overall term [17]. Moreover, since this factorization does not rely on chiral counting, we are lead to write (cf. Eq. (12)):

$$f_+(t, u) = f_+(t)\Big[1 + f_{\text{loop}}^{\text{elm}}(u, M_\gamma)\Big] \tag{20}$$

$$f_+(t) = \frac{M_\rho^2}{M_\rho^2 - t - iM_\rho\Gamma_\rho(t)}\exp\Big[2\tilde{H}_{\pi^0\pi^-}(t) + \tilde{H}_{K^0 K^-}(t)\Big] + f_{\text{local}}^{\text{elm}} \; . \tag{21}$$

As in the case of $F_V(t)$ in (12), this representation of $f_+(t)$ has the correct low-energy behaviour to $O(p^4)$ and it interpolates smoothly to the resonance region. The resonance width $\Gamma_\rho(t)$ in (13) has to be calculated now with the appropriate $\pi^-\pi^0$ and K^-K^0 thresholds and phase space factors.

The photon loop amplitude $f_{\text{loop}}^{\text{elm}}(u, M_\gamma)$ is infrared divergent depending on an artificial photon mass M_γ. This dependence is canceled by bremsstrahlung of soft photons making the decay distribution in (t, u) infrared finite. The sum of real and virtual contributions produces the following correction factor to the (t, u) decay distribution

$$\Delta(t, u) = 1 + 2f_{\text{loop}}^{\text{elm}}(u, M_\gamma) + g_{\text{brems}}(t, u, M_\gamma, E_\gamma^{\text{min}}) \; , \tag{22}$$

which depends on the minimal photon energy $E_\gamma^{\min}$ detected in the apparatus and is independent of M_γ. This factor has to be multiplied by the kinematical density $D(t, u)$ (defined in the Appendix) and integrated over the variable u to produce the term $G_{\mathrm{EM}}(t)$ in the decay distribution (4) with respect to t:

$$G_{\mathrm{EM}}(t) = \frac{\displaystyle\int_{u_{\min}(t)}^{u_{\max}(t)} du\, D(t, u)\, \Delta(t, u)}{\displaystyle\int_{u_{\min}(t)}^{u_{\max}(t)} du\, D(t, u)} \; . \tag{23}$$

The details of soft photon emission (and the function $g_{\mathrm{brems}}(t, u, M_\gamma, E_\gamma^{\min})$) depend on the specific experimental setup. To the best of our knowledge, all τ decay experiments relevant here [3] apply bremsstrahlung corrections in the same (approximate) way described in Ref. [18]: only the leading term in the Low expansion (proportional to $1/E_\gamma$ in the amplitude) is taken into account including also the logarithmic term in the loop amplitude $f_{\mathrm{loop}}^{\mathrm{elm}}(u, M_\gamma)$ depending on M_γ (contained in the function $\mathcal{C}(u, M_\gamma)$ given in the Appendix). As emphasized in Ref. [18], this is only an approximate treatment of bremsstrahlung that can be trusted for sufficiently small $E_\gamma^{\min}$. Assuming this prescription, the setup-independent part of $\Delta(t, u)$ therefore involves only the subtracted loop amplitude (x is defined in the Appendix)

$$f_{\mathrm{loop,sub}}^{\mathrm{elm}}(u) = f_{\mathrm{loop}}^{\mathrm{elm}}(u, M_\gamma) + \frac{\alpha}{2\pi}\, (M_\pi^2 + m_\tau^2 - u) \frac{1}{m_\tau M_\pi} \frac{x}{1 - x^2} \log x \log \frac{M_\gamma^2}{m_\tau M_\pi} \tag{24}$$

replacing $f_{\mathrm{loop}}^{\mathrm{elm}}(u, M_\gamma)$ in (22).

Following Ref. [17], we have factored out a universal loop amplitude $f_{\mathrm{loop,sub}}^{\mathrm{elm}}(u)$. Although this factorization is independent of the low-energy expansion it is interesting to analyse the dependence on the lepton mass m_τ. Unlike in K_{l3} decays [14], the charged lepton is not light compared to a typical hadronic scale $\sim M_\rho$. However, we can perform an expansion in p/m_τ in complete analogy to heavy baryon CHPT where p stands for a typical meson mass or momentum. Expanding $f_{\mathrm{loop,sub}}^{\mathrm{elm}}(u)$ in inverse powers of m_τ yields

$$f_{\mathrm{loop,sub}}^{\mathrm{elm}}(u) = \frac{\alpha}{4\pi}\left(-1 + \log\frac{m_\tau^2}{M_\pi^2} + O(\frac{p}{m_\tau})\right) \; . \tag{25}$$

It turns out that the function $G_{\mathrm{EM}}(t)$ is quite insensitive to whether it is calculated from the full $f_{\mathrm{loop,sub}}^{\mathrm{elm}}(u)$ in (24) or from its large-m_τ approximation (25). The difference is negligible in the full range $4M_\pi^2 \leq t \leq 0.8$ GeV^2: the leading term in (25) provides an excellent approximation.

5. The results of our analysis are summarized in Figs. 2 (a),(b) where we plot the function $R_{\mathrm{IB}}(t)$ and its component factors defined in Eq. (9) for $0.2 \leq t \leq 0.8$ GeV^2. We note that the dominant contribution at low t is given by the kinematical term $\beta_{+-}^3/\beta_{0-}^3$ [10]. Photonic corrections embodied in $G_{\mathrm{EM}}(t)$ reduce $R_{\mathrm{IB}}(t)$ in addition by about half a
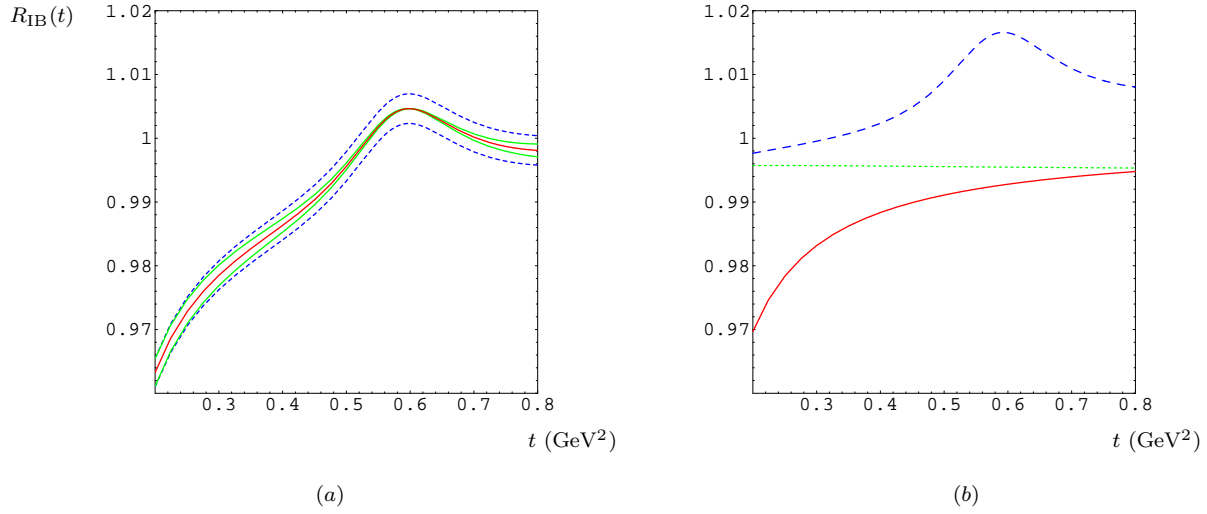
Figure 2: (*a*) Correction factor $R_{\text{IB}}(t)$ for isospin violation. The bands around the central curve correspond to the uncertainty in the low-energy constants (solid lines) and in the bremsstrahlung factor (dashed lines). (*b*) The separate factors defining $R_{\text{IB}}(t)$ in Eq.(9) are plotted as solid line for $\beta_{+-}^3/\beta_{0-}^3$, dashed line for $|F_V(t)/f_+(t)|^2$ and dotted line for $1/G_{\text{EM}}(t)$.

percent, largely independently of t. The form factor ratio $|F_V(t)/f_+(t)|^2$ is dominated by the width difference $\Gamma_{\rho^+} - \Gamma_{\rho^0}$.

We have used the following input for the calculation of $R_{\text{IB}}(t)$.

- We employ the form factor $F_V(t)$ given in Eq. (12).

- We use the form factor $f_+(t)$ as given in Eq. (21). The local contribution depends on three low-energy constants appearing in the chiral expansion. For the constant $K_{12}^r(\mu)$ a sum rule representation is available [19]. Saturating the sum rule with low-lying resonances and choosing the QED renormalization scale between 0.5 and 1 GeV, we arrive at the following estimate:

$$K_{12}^r(M_\rho) = -(3 \pm 1) \times 10^{-3} \ .$$

As for X_1 and $\tilde{X}_6^r(\mu)$, no estimates are presently available. We therefore use the upper bounds suggested by dimensional analysis:

$$|X_1| \leq \frac{1}{(4\pi)^2} \ , \quad |\tilde{X}_6^r(M_\rho)| \leq \frac{5}{(4\pi)^2} \ .$$

In the case of $\tilde{X}_6^r(M_\rho)$, we have enlarged the naive estimate by a factor of 5 (the β function associated with X_6 [16]). This accounts for the present uncertainty in the

8

matching to the short-distance contribution to X_6 performed in Eqs. (18, 19). The corresponding uncertainty in $R_{\mathrm{IB}}(t)$ is shown in Fig. 2 (a) (solid curves).

- We include the factor $G_{\mathrm{EM}}(t)$ according to the discussion following Eq. (22). Due to neglect of sub-leading terms [18] in the function $g_{\mathrm{brems}}(t, u, M_\gamma, E_\gamma^{\min})$, $G_{\mathrm{EM}}(t)$ can receive extra contributions of order $\alpha/(4\pi) \times O(1)$. We therefore assign an uncertainty of $\pm\alpha/\pi$ to it. The effect on $R_{\mathrm{IB}}(t)$ is also shown in Fig. 2(a) (dashed curves). Clearly, better knowledge of the radiative amplitude can be used to improve our determination of $G_{\mathrm{EM}}(t)$.

In order to quantify the impact of $R_{\mathrm{IB}}(t)$ on a_μ^{vacpol}, we construct the following ratio:

$$\mathcal{R}(t_{\max}) = \frac{\int_{4M_\pi^2}^{t_{\max}} dt\, K(t)\, \sigma_{e^+e^-\to\pi^+\pi^-}^{0,\mathrm{CVC}}(t)\, R_{\mathrm{IB}}(t)}{\int_{4M_\pi^2}^{t_{\max}} dt\, K(t)\, \sigma_{e^+e^-\to\pi^+\pi^-}^{0,\mathrm{CVC}}(t)} \ , \tag{26}$$

where $\sigma_{e^+e^-\to\pi^+\pi^-}^{0,\mathrm{CVC}}(t)$ is obtained via CVC from the τ decay distribution as given in Eq. (7). A few representative values of $\mathcal{R}(t_{\max})$ are given in Table 1. To translate the ratio $\mathcal{R}(t_{\max})$ into a modification of $a_\mu^{\mathrm{vacpol}}(4M_\pi^2 \leq t \leq t_{\max})$, we take for the purpose of illustration the value $a_\mu^{\mathrm{vacpol}}(4M_\pi^2 \leq t \leq 0.8\ \mathrm{GeV}^2) = (4794.6 \pm 60.7) \times 10^{-11}$ of Ref. [1]. Let us assume for simplicity that this value is calculated from τ decay data only. As the quoted number contains the ratio of the radiatively corrected hadronic rate and the measured electronic mode (including the radiative channel with a photon in the final state), we have to multiply it [20] by the correction factor $1 + (\alpha/2\pi)(25/4 - \pi^2)$ of Eq. (5) in addition to $\mathcal{R}(t_{\max} = 0.8\ \mathrm{GeV}^2)$. In this way, isospin violation would reduce $a_\mu^{\mathrm{vacpol}}(4M_\pi^2 \leq t \leq 0.8\ \mathrm{GeV}^2)$ by 76×10^{-11} or about one standard deviation of the reported error.

Table 1: Correction factor for a_μ^{vacpol} due to isospin violation (defined in Eq. (26)) for some values of $t_{\max}$. An uncertainty of 0.002 - reflecting the one in the bremsstrahlung factor $G_{\mathrm{EM}}(t)$ - should be assigned to the values reported here. This is also an upper bound for the uncertainty due to the low-energy constants (see Fig. 2(a)).

$t_{\max}$ (GeV^2)	0.3	0.5	0.8
$\mathcal{R}(t_{\max})$	0.949	0.974	0.988

6. We have calculated the leading isospin-breaking corrections to the CVC relation between the $e^+e^- \to \pi^+\pi^-$ cross section and the decay distribution for $\tau^- \to \pi^0\pi^-\nu_\tau$. The calculation was performed in the framework of CHPT to $O[(m_u - m_d)p^2)]$ and $O(e^2p^2)$.

Our main result comes in the form of a function $R_{\mathrm{IB}}(t)$ displayed in Fig. 2(a) that corrects the CVC relation for isospin violation and electromagnetic effects. Since $R_{\mathrm{IB}}(t)$ is smaller than unity in most of the region under consideration ($4M_\pi^2 \leq t \leq 0.8$ GeV^2) isospin violation accounts at least for a sizable part of the systematic difference at low energies between e^+e^- and τ decay data (e.g., Ref. [21]).

In general, isospin-violating corrections are expected to be of the order

$$\frac{\Delta_\pi}{M_\rho^2} = 2 \times 10^{-3} \tag{27}$$

where M_ρ stands for a typical hadronic scale and $\Delta_\pi = M_{\pi^+}^2 - M_{\pi^0}^2$. Electromagnetic corrections embodied in the function $G_{\mathrm{EM}}(t)$ are precisely of this magnitude as shown in Fig. 2(b). In the form factor ratio $|F_V(t)/f_+(t)|^2$, the ratio (27) is enhanced by a numerical factor:

$$\left|\frac{F_V(M_\rho^2)}{f_+(M_\rho^2)}\right|^2 \simeq \frac{\Gamma_{\rho^+}^2}{\Gamma_{\rho^0}^2} \simeq 1 + \frac{6\Delta_\pi}{M_\rho^2 - 4M_\pi^2} = 1.015 \ . \tag{28}$$

The biggest effect occurs in the phase space ratio $\beta_{+-}^3/\beta_{0-}^3$. It is governed by the function

$$\frac{3\Delta_\pi}{t - 4M_\pi^2} \tag{29}$$

and therefore dominates at low energies.

Although the calculation is based on a low-energy effective description of the standard model we claim that the main features of the correction factor $R_{\mathrm{IB}}(t)$ are valid up to $t \simeq 0.8$ GeV^2. Of the three factors in the definition (9) of $R_{\mathrm{IB}}(t)$, both the dominant phase space correction factor [10] and the photon loop effects are independent of the low-energy expansion. Finally, the main part of isospin violation in the form factor ratio $|F_V(t)/f_+(t)|^2$ occurs in the ρ-width difference $\Gamma_{\rho^+} - \Gamma_{\rho^0}$ and should therefore be reliable in the vicinity of the resonance.

In this work we have been concerned with corrections to the CVC relation between τ data and the *bare* cross section $\sigma^0_{e^+e^- \to \pi^+\pi^-}$. An important next-to-leading-order effect of $O(\alpha^3)$ in a_μ^{had} involves final state radiative corrections in $\sigma_{e^+e^- \to \pi^+\pi^-}$. Within scalar QED, this was already calculated in Ref. [22] and recently reported in [20]. At $O(e^2p^2)$ in CHPT one obtains the same result [23] as in scalar QED because the local counterterm contributions cancel due to gauge invariance. The resulting correction to a_μ^{had} of $O(\alpha^3)$ is positive [20].

Appendix The loop function $h_{PQ}(t, \mu)$ is given by

$$
\begin{aligned}
h_{PQ}(t, \mu) &= \frac{1}{12t} \lambda(t, M_P^2, M_Q^2)\, \bar{J}^{PQ}(t) + \frac{1}{18(4\pi)^2}(t - 3\Sigma_{PQ}) \\
&\quad - \frac{1}{12}\left(\frac{2\Sigma_{PQ} - t}{\Delta_{PQ}}(A_P(\mu) - A_Q(\mu)) - 2(A_P(\mu) + A_Q(\mu))\right),
\end{aligned} \tag{30}
$$

where

$$
\Sigma_{PQ} = M_P^2 + M_Q^2, \qquad \Delta_{PQ} = M_P^2 - M_Q^2
$$

$$
A_P(\mu) = -\frac{M_P^2}{(4\pi)^2} \log \frac{M_P^2}{\mu^2}
$$

$$
\begin{aligned}
\bar{J}^{PQ}(t) &= \frac{1}{32\pi^2}\left[2 + \frac{\Delta_{PQ}}{t} \log \frac{M_Q^2}{M_P^2} - \frac{\Sigma_{PQ}}{\Delta_{PQ}} \log \frac{M_Q^2}{M_P^2}\right. \\
&\quad \left. - \frac{\lambda^{1/2}(t, M_P^2, M_Q^2)}{t} \log\left(\frac{(t + \lambda^{1/2}(t, M_P^2, M_Q^2))^2 - \Delta_{PQ}^2}{(t - \lambda^{1/2}(t, M_P^2, M_Q^2))^2 - \Delta_{PQ}^2}\right)\right].
\end{aligned} \tag{31}
$$

In terms of the variables

$$
r_\tau = \frac{m_\tau^2}{M_\pi^2}, \qquad y = 1 + r_\tau - \frac{u}{M_\pi^2}, \qquad x = \frac{1}{2\sqrt{r_\tau}}(y - \sqrt{y^2 - 4r_\tau}), \tag{32}
$$

and of the dilogarithm

$$
Li_2(x) = -\int_0^1 \frac{dt}{t} \log(1 - xt), \tag{33}
$$

the functions contributing to $f_{\text{loop}}^{\text{elm}}(u, M_\gamma)$ are given by

$$
\mathcal{A}(u) = \frac{1}{u}\left[-\frac{1}{2} \log r_\tau + \frac{2 - y}{\sqrt{r_\tau}} \frac{x}{1 - x^2} \log x\right] \tag{34}
$$

$$
\mathcal{B}(u) = \frac{1}{u}\left[\frac{1}{2} \log r_\tau + \frac{2r_\tau - y}{\sqrt{r_\tau}} \frac{x}{1 - x^2} \log x\right] \tag{35}
$$

$$
\begin{aligned}
\mathcal{C}(u, M_\gamma) &= \frac{1}{m_\tau M_\pi} \frac{x}{1 - x^2}\left[-\frac{1}{2} \log^2 x + 2 \log x \log(1 - x^2) - \frac{\pi^2}{6} + \frac{1}{8} \log^2 r_\tau\right. \\
&\quad \left. + Li_2(x^2) + Li_2\left(1 - \frac{x}{\sqrt{r_\tau}}\right) + Li_2(1 - x\sqrt{r_\tau}) - \log x \log \frac{M_\gamma^2}{m_\tau M_\pi}\right]. \tag{36}
\end{aligned}
$$

The kinematical weight to be used in Eq. (23) is

$$
D(t, u) = \frac{m_\tau^2}{2}(m_\tau^2 - t) + 2M_{\pi^0}^2 M_{\pi^-}^2 - 2\,u\,(m_\tau^2 - t + M_{\pi^0}^2 + M_{\pi^-}^2) + 2\,u^2. \tag{37}
$$

The integration limits in Eq. (23) are given by

$$
u_{\text{max/min}}(t) = a(t) \pm b(t) \tag{38}
$$

$$
a(t) = \frac{1}{2}\left[-\Delta_\pi\left(1 + \frac{m_\tau^2}{t}\right) + 2M_{\pi^-}^2 + m_\tau^2 - t\right] \tag{39}
$$

$$
b(t) = \frac{1}{2}\left(m_\tau^2 - t\right) \lambda^{1/2}\left(1, \frac{M_{\pi^+}^2}{t}, \frac{M_{\pi^0}^2}{t}\right). \tag{40}
$$

11

References

[1] S. Narison, `hep-ph/0103199`.

[2] R. Alemany, M. Davier and A. Höcker, Eur. Phys. J. **C2** (1998) 123;
M. Davier and A. Höcker, Phys. Lett. **B435** (1998) 427.

[3] R. Barate et al. (ALEPH), Eur. Phys. J. **C4** (1998) 409;
K. Ackerstaff et al. (OPAL), Eur. Phys. J. **C7** (1999) 571;
S. Anderson et al. (CLEO), Phys. Rev. **D61** (2000) 112002.

[4] H.N. Brown et al. (BNL-E821), Phys. Rev. Lett. **86** (2001) 2227.

[5] S. Weinberg, Physica **96A** (1979) 327;
J. Gasser and H. Leutwyler, Ann. Phys. **158** (1984) 142;
J. Gasser and H. Leutwyler, Nucl. Phys. **B250** (1985) 465;
H. Leutwyler, Ann. Phys. **235** (1994) 165.

[6] M. Gourdin and E. de Rafael, Nucl. Phys. **B10** (1969) 667.

[7] S. Eidelman and F. Jegerlehner, Z. Phys. **C67** (1995) 585.

[8] W.J. Marciano and A. Sirlin, Phys. Rev. Lett. **61** (1988) 1815; ibid. **71** (1993) 3629.

[9] T. Kinoshita and A. Sirlin, Phys. Rev. **113** (1959) 1652.

[10] H. Czyz and J.H. Kühn, Eur. Phys. J. **C18** (2001) 497;
J.H. Kühn, `hep-ph/0101100`.

[11] J. Gasser and H. Leutwyler, Nucl. Phys. **B250** (1985) 517.

[12] F. Guerrero and A. Pich, Phys. Lett. **B412** (1997) 382;
D. Gómez Dumm, A. Pich and J. Portolés, Phys. Rev. **D62** (2000) 054014.

[13] K. Maltman, Phys. Rev. **D58** (1998) 014008.

[14] V. Cirigliano et al., in preparation.

[15] R. Urech, Nucl. Phys. **B433** (1995) 234;
H. Neufeld and H. Rupertsberger, Z. Phys. **C68** (1995) 91; ibid. **C71** (1996) 131.

[16] M. Knecht, H. Neufeld, H. Rupertsberger and P. Talavera, Eur. Phys. J. **C12** (2000) 469.

[17] D. R. Yennie, S. C. Frautschi and H. Suura, Ann. Phys. **13** (1961) 379.

[18] E. Barberio and Z. Wąs, Comp. Phys. Comm. **79** (1994) 291.

[19] B. Moussallam, Nucl. Phys. **B504** (1997) 381.

[20] K. Melnikov, `hep-ph/0105267` .

[21] S.I. Eidelman and V.N. Ivanchenko, Nucl. Phys. B (Proc. Suppl.) **76** (1999) 319.

[22] J. Schwinger, in *Particles, Sources and Fields*, vol. II, Perseus Books, Reading, Massachusetts (1989);
M. Drees and K. Hikasa, Phys. Lett. **B252** (1990) 127.

[23] B. Kubis and U.-G. Meißner, Nucl. Phys. **A671** (2000) 332, `hep-ph/9908261`.